1 Lanchester Model

The equations for the Lanchester model of battles is

$$\begin{aligned} A' &= -\beta B \\ B' &= -\alpha A. \end{aligned} \tag{1}$$

The initial conditions of the model are A_0 and B_0, with $x_0 = A_0/B_0$.

The time derivative of $x = A/B$ is

$$d(A/B)/dt = A'/B - B'A/B^2. \tag{2}$$

Substituting A' and B' from Equation 1 yields

$$dx/dt = -\beta + \alpha x^2. \tag{3}$$

Note that if the initial conditions are such that $dx/dt < 0$, this implies that x is decreasing, and $\alpha x^2 < \beta$. Thus when dx/dt is initially less than zero, the LHS of that inequality becomes increasinging smaller, and thus dx/dt becomes increasingly negative, and x monotonically decreases. Similarly, if the initial conditions are such that $dx/dt > 0$, this implies that x is increasing, and $\alpha x^2 > \beta$. Thus the dx/dt becomes increasingly positive, and x monotonically increases. Thus, for the deterministic model as written, it is pre-destined from the beginning of the battle which army will win.

Re-arranging Equation 3 yields

$$\begin{aligned} dt &= \frac{dx}{-\beta + \alpha x^2} \\ &= -\frac{1}{\alpha}\frac{dx}{D^2 - x^2}, \end{aligned} \tag{4}$$

with $D = \sqrt{\beta/\alpha}$. Integrating both sides yields two solutions, one when $x_0 < D$, and the other when $x_0 > D$.

When $x_0 < D$, B wins, and we have

$$t + C = \frac{\tanh^{-1} x/D}{\alpha D}, \tag{5}$$

where C is

$$C = \frac{1}{\alpha D}\tanh^{-1} x_0/D. \tag{6}$$

The time at which B wins is

$$t_{\text{win}} = \frac{\tanh^{-1} 0.1/D}{\alpha D} - C. \tag{7}$$

When $x_0 > D$, A wins, and we have

$$t + C = \frac{1}{2\alpha D}\log\left(\frac{(x - D)}{(x + D)}\right), \tag{8}$$

where C is

$$C = \frac{1}{2\alpha D} \log\left(\frac{(x_0 - D)}{(x_0 + D)}\right), \tag{9}$$

The time at which A wins is

$$t = \frac{1}{2\alpha D} \log\left(\frac{(10 - D)}{(10 + D)}\right) - C. \tag{10}$$